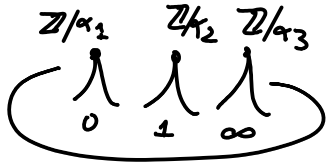
A. Takahashi – 27/5/08 – HMS for isolated hypersurface singularities

---

<u>Goal</u>: HMS for "cusp" singularities

- $A = (\alpha_1, \alpha_2, \alpha_3) \qquad \alpha_i \in \mathbb{Z}_{>0}$

$\rightsquigarrow \qquad X_A =$ 

$\mathbb{P}^1$-orbifold

$$R_A = \mathbb{C}[x,y,z] \,/\, (x^{\alpha_1} - y^{\alpha_2} + z^{\alpha_3})$$
$$L_A := \mathbb{Z}\vec{x} + \mathbb{Z}\vec{y} + \mathbb{Z}\vec{z} \,/\, (\alpha_1\vec{x} = \alpha_2\vec{y} = \alpha_3\vec{z})$$

$$D^b \operatorname{coh} X_A := D^b \operatorname{gr}^{L_A}\text{-}R_A \,/\, D^b(\operatorname{tor} R_A)$$

L_A-graded R_A-modules mod. torsion modules

<u>Geigle–Lenzing</u>: $D^b \operatorname{coh} X_A \simeq D^b \operatorname{mod} k\vec{Q}/I$

where quiver $\vec{Q} =$

$$\begin{tikzcd}[column sep=small]
 & \mathcal{O}(\vec{x}) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((\alpha_1-1)\vec{x}) \arrow[dr] & \\
\mathcal{O} \arrow[ur, "x"] \arrow[r, "y"] \arrow[dr, "z"'] & \mathcal{O}(\vec{y}) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((\alpha_2-1)\vec{y}) \arrow[r] & \mathcal{O}(\alpha_1\vec{x}) \\
 & \mathcal{O}(\vec{z}) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((\alpha_3-1)\vec{z}) \arrow[ur] &
\end{tikzcd}$$

$$I = (x^{\alpha_1} - y^{\alpha_2} + z^{\alpha_3})$$

- <u>Conj</u>: $D^b \operatorname{coh} X_A \simeq D^b \operatorname{Fuk}_{gl}(f_A)$ if $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \geq 1 \quad (*)$

  $D^b \operatorname{Fuk}_0(f_A)$ if $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} < 1. \quad (**)$

  restriction to contribution of singularity at 0, i.e. truncate to $B(0,\varepsilon) \subset \mathbb{C}^3$

  where $f_A = x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_3} + xyz \; : \; \mathbb{C}^3 \to \mathbb{C}$

  or $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$

(Fukaya categories of LG model – cf. Seidel: gen.d by vanishing cycles)

NB: case $(*)$: $f_A : \mathbb{C}^3 \to \mathbb{C}$ has $\dim_{\mathbb{C}} \left(\mathbb{C}[x,y,z]/(\partial f) \right) = \mu = \alpha_1+\alpha_2+\alpha_3-1$

case $(**)$: $f_A : (\mathbb{C}^3,0) \to (\mathbb{C},0)$ has $\dim_{\mathbb{C}} \left(\mathbb{C}[[x,y,z]]/(\partial f) \right) =$

Milnor number of the singularity

NB Seidel: for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, should get usual $\mathbb{P}^1$...
here we have $f = x+y+z+xyz$ on $\mathbb{C}^3$
is stably equivalent to usual mirror $(\mathbb{P}^1)$

$$\left(f = x - \frac{1}{x} + (xy+1)\left(z+\frac{1}{x}\right) \text{ e.g.} \right)$$

- Case $(*)$ is: $\begin{cases} \frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3} > 1 & \leftrightarrow \text{ ADE sing.} \\ \frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3} = 1 & \leftrightarrow \text{ simple elliptic singularity} \end{cases}$

in particular $\begin{cases} (p,q,1) \leftrightarrow A_{p+q} \\ (n,2,2) \leftrightarrow D_{n+2} \\ (3,3,2) \leftrightarrow E_6 \\ (4,3,2) \leftrightarrow E_7 \\ (5,3,2) \leftrightarrow E_8 \end{cases}$

- Case $(\alpha_1, \alpha_2, \alpha_3) = (p,q,1)$ of the conjecture is proved by Auroux - Katzarkov - Orlov

Rmk: If conj. for $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3} > 1$ is true then

expect $D^b_E \operatorname{Coh}(\tilde{X}_A) \underset{(?)}{\xrightarrow{\sim}} D^b \operatorname{Fuk}(Y)$

resolution of ADE sing. $\qquad$ $Y =$ Milnor fiber of f_A.

$$D^b\mathrm{Fuk}(f+x^2) \overset{(?)}{\xrightarrow{\sim}} D^b\mathrm{Fuk}(f)$$

$\rightsquigarrow$ let $\underline{g_A := x^{a_1} + y^{a_2} - x^2y^2}$

(for case $n=2$: stabilize $g_A + (xy+z)^2 \simeq f_A$ ✓)

<u>Thm:</u> $D^b\mathrm{coh}\, X_A \cong D^b\mathrm{Fuk}(g_A)$ in cases $\tilde{D}_{2n+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

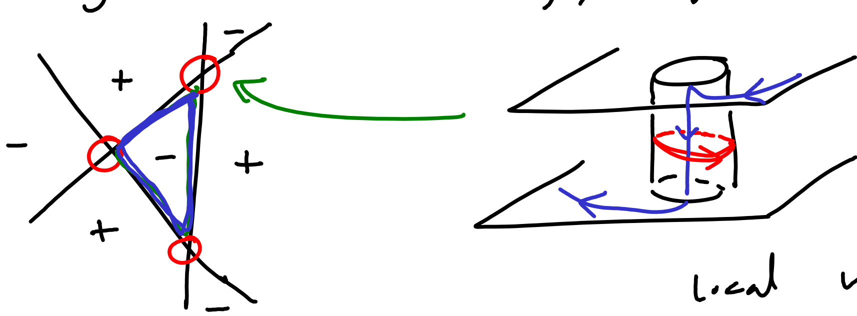
($\sim$ extended Dynkin diagrams)

<u>key idea:</u> use a "real Morsification" of g_A and A'Campo's description of vanishing cycles (cf. Seidel's "more about VCM")

$\tilde{g}_A$ = real polynomial s.t. $\tilde{g}_A = \tilde{g}_1 \cdots \tilde{g}_r$

where $\tilde{g}_A = 0$ has δ ODP's in $\mathbb{R}^2$ $\quad (\mu = 2\delta - r + 1)$

<u>Ex:</u> $f = x^3 - xy^2 \rightsquigarrow \tilde{f} = (x^2-y^2)(x-1)$

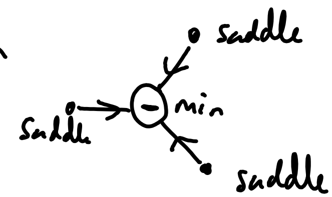


local vanishing cycles

$\begin{cases} \text{1 vc for each saddle} & \text{(real Morse index 1} \\ \text{1 vc for each bounded } +/- \text{ region} & \text{0,2)} \end{cases}$

Intersections b/w these vc's $\Leftrightarrow$ real gradient flow lines of $\tilde{f}_{\mathbb{R}} \ldots$

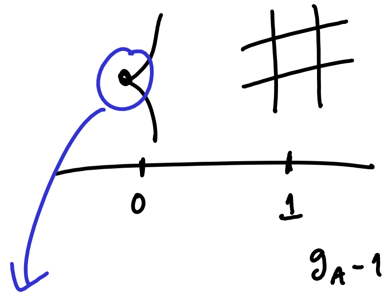
e.g. here get diagram



for quiver

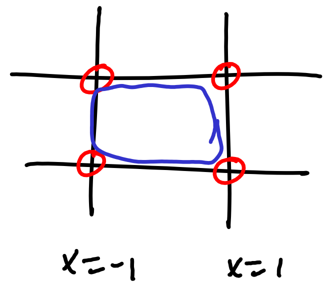
Ex: $\alpha_1 = \alpha_2 = 2$: $\quad g_A = x^2+y^2-x^2y^2$

$$g_A : \begin{array}{c}\mathbb{C}^2\\ \downarrow\\ \mathbb{C}\end{array}$$

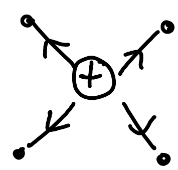


$$g_A - 1 = (x^2-1)(y^2-1)$$

Real Morsification ...

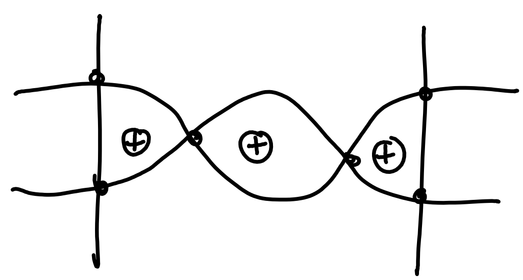


gives

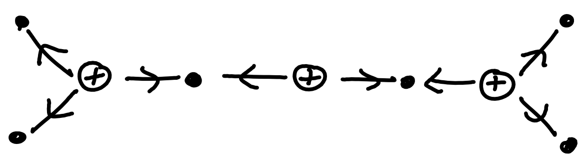


$\tilde{D}_4$

Ex: $x^4+y^2-x^2y^2$:



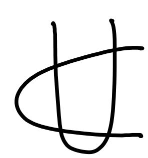
$\rightsquigarrow$



$\tilde{D}_8$

Ex: $x^3+y^3-x^2y^2$

$\rightsquigarrow \tilde{g} = (x-y^2+c)(y-x^2+c')$

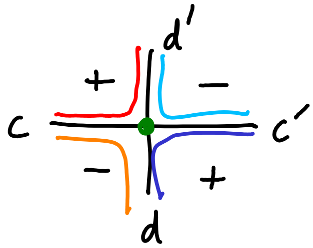


get $\tilde{E}_6$

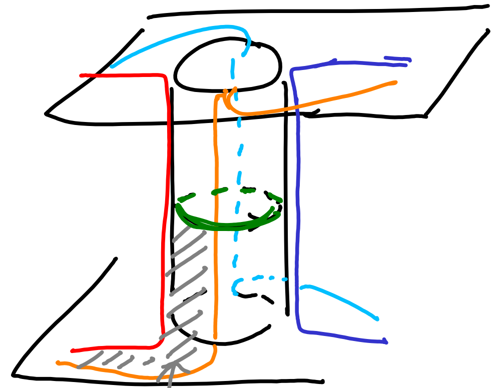
Similarly $\tilde{E}_7$, $\tilde{E}_8$ ... $\Rightarrow$ get in all cases $D^b\mathrm{Fuk}(g_A) \simeq D^b\mathrm{mod}(k\vec{\tilde{\Delta}}_A)$

Dynkin diagram

Rmk:



$\Longleftrightarrow$



only nontrivial m_k's are of this type.

$\Rightarrow$ $\overrightarrow{\mathrm{Fuk}}$ has no m_k for $k \neq 2$

then check commutation relation .... ✓